DOI:10.13471/j.cnki.acta.snus.2016.06.012

基于 Caputo 导数下的含时滞的 Hamilton 系统的分数阶 Noether 理论*

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摘　要：提出并讨论了 Caputo 导数定义下的含时滞的 Hamilton 系统的分数阶 Noether 对称性与守恒量。根据含时滞的 Hamilton 系统的分数阶 Hamilton 原理，建立了相应的含时滞的分数阶 Hamilton 正则方程；依据分数阶 Hamilton 作用量在无限小变换下的不变性，得到了含时滞的 Hamilton 系统的分数阶 Noether 对称性；最后，建立了系统的含时滞的分数阶 Noether 理论，并举例说明结果的应用。

关键词：时滞；Hamilton 系统；Caputo 导数；Noether 对称性；守恒量

中图分类号：O316　　**文献标志码：**A　　**文章编号：**0529-6579（2016）06-0079-07

Fractional Noether theorems for Hamilton system with time delay based on Caputo dervitaves

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Abstract: The fractional Noether symmetries and fractional conserved quantities for Hamilton system with time delay based on Caputo derivatives are discussed. The fractional Hamilton canonical equations of the corresconding system with time delay are established base upon the fractional Hamilton principle of the Hamilton systems with time delay. Then, the fractional Noether symmetries of the Hamilton system with time delay are obtained, which based on the invariance of the fractional Hamilton action with time delay under the infinitesimal transformations of group. Finally, fractional Noether theorems with time delay of the Hamilton system are established. At the end, one example is given to illustrate the application of the results.

Key words: time delay; Hamilton system; Caputo derivatives; Noether symmetry; conserved quantity

随着人们认识的深入，发现自然界在本质上是分数阶的，使用分数阶模型能够更好地描述和理解复杂系统的动力学行为及其内在的物理本质，因此分数阶微积分开始广泛应用于力学、物理学、生物学、化学、经济和社会科学，智能控制等领域[1-2]。1996 年，Riewe[3-4] 首次将分数阶微积分应用到非保守力学建模，开始对于分数阶变分问题的研究近 20 年来，关于分数阶模型下的变分问题

* **收稿日期：**2016-05-12
基金项目：国家自然科学基金资助项目（10972151，11272227）
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的研究取得了一系列成果[5-11]。Frederico 和 Torres[12-13]研究了分数阶模型下的分数阶变分与最优化问题的 Noether 定理。Atanackovi 等[14]依据经典的守恒量定义了分数阶守恒量，得到了力学系统的分数阶 Noether 定理。然而对于分数阶导数下含时滞的变分问题的研究是最近几年才开始的。2008 年，Baleanu，Maaraba 和 Jarad[15-17]首次讨论了分数阶导数下的含时滞的分数阶变分问题，又研究了 Caputo 分数阶导数下的含时滞的分数阶变分原理，并讨论了含时滞的 Caputo 分数阶导数下的高阶分数阶变分最优化控制问题。张毅等[18-21]对分数阶导数下非保守力学以及 Birkhoff 系统动力学的变分问题及其对称性进行了研究，并得到了一些研究成果。本文将进一步研究基于 Caputo 导数的含时滞的 Hamilton 系统动力学的分数阶 Noether 对称性。

1 分数阶导数的定义及其性质

本节列出将要用到的 Riemann-Liouville 导数以及 Caputo 导数的定义和性质。详细讨论见文献［1-2］。

左 Riemann-Liouville 分数阶导数的定义为[5]

$$ {}_{t_1}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \cdot \int_{t_1}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau \tag{1} $$

右 Remiann-Liouville 分数阶导数的定义为

$$ {}_tD_{t_2}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \cdot \int_{t}^{t_2}(\tau-t)^{n-\alpha-1}f(\tau)\mathrm{d}\tau \tag{2} $$

左 Caputo 分数阶导数为

$$ {}_{t_1}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{t_1}^{t}(t-\tau)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(\tau)\mathrm{d}\tau \tag{3} $$

右 Caputo 分数阶导数为

$$ {}_{t}^{C}D_{t_2}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{t}^{t_2}(\tau-t)^{n-\alpha-1}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(\tau)\mathrm{d}\tau \tag{4} $$

其中 $\Gamma(*)$ 为 Gamma 函数，α 为导数的阶，满足 $n-1 \leqslant \alpha < n$。若 α 为整数时，有

$$ {}_{t_1}D_t^{\alpha}f(t) = {}_{t_1}^{C}D_t^{\alpha}f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}f(t), $$

$$ {}_tD_{t_2}^{\alpha}f(t) = {}_{t}^{C}D_{t_2}^{\alpha}f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}f(t) \tag{5} $$

分数阶分部积分公式为

$$ \int_{t_1}^{r} g t_{t_1}D_t^{\alpha}f(t)\mathrm{d}t = \int_{t_1}^{r} f(t)\,{}_tD_r^{\alpha}g(t)\mathrm{d}t \tag{6} $$

以及

$$ \int_{r}^{t_2} g(t)\,{}_{t_1}D_t^{\alpha}f(t)\mathrm{d}t = \int^{t_2} f(t)\,{}_{t_1}D_{t_2}^{\alpha}g(t)\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r}{}_{t1}D_t^{\alpha}f(t)\left[\int_{t}^{t_2}D_{t_2}^{\alpha}g(z)(z-t)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}t = \int^{t_2} f(t)\,{}_tD_{t_2}^{\alpha}g(t)\mathrm{d}t - \frac{1}{\Gamma(\alpha)} \cdot \int_{t_1}^{r} f(t)\,{}_tD_r^{\alpha}\left[\int_{t}^{t_2}D_{t_2}^{\alpha}g(z)(z-t)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}t \tag{7} $$

其中 $r \in (t_1, t_2)$，且有

$$ \int_{t_1}^{t_2} g(t)\,{}_{t_1}^{C}D_t^{\alpha}f(t)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\,{}_tD_{t_2}^{\alpha}g(t)\mathrm{d}t + \sum_{j=0}^{n-1}{}_tD_{t+j-n_2}^{\alpha}g(t)D^{n-1-j}f(t)\Big|_{t_1}^{t_2} \tag{8} $$

$$ \int_{t_1}^{t_2} g(t)\,{}_{t}^{C}D_{t_2}^{\alpha}f(t)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\,{}_{t_1}D_t^{\alpha}g(t)\mathrm{d}t + \sum_{j=0}^{n-1}(-1)^{n+j}_{t}D_{t+j-n_2}^{\alpha}g(t)D^{n-1-j}f(t)\Big|_{t_1}^{t_2} \tag{9} $$

2 含时滞的分数阶 Hamilton 正则方程

假设力学系统由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 来确定。含时滞的分数阶 Lagrange 函数为

$$ L = L(t, q_s(t), {}_{t_1}^{C}D_t^{\alpha}q_s(t), \dot{q}_s(t), q_{s\tau}(t), \dot{q}_{s\tau}(t)) = L(t, q_s, {}_{t_1}^{C}D_t^{\alpha}q_s, \dot{q}_s(t), q_{s\tau}, \dot{q}_{s\tau}) \tag{10} $$

其中时滞常量 $\tau < t_2 - t_1$ 为已知的正实数。

引入含时滞的分数阶广义动量和 Hamilton 函数为

$$ p_{\alpha s}(t) = \frac{\partial L}{\partial\, {}_{t_1}^{C}D_t^{\alpha}q_s}(t) p_s(t) = \frac{\partial L}{\partial \dot{q}_s}(t) p_{s\tau} = p_{s\tau}(t) = \frac{\partial L}{\partial \dot{q}_{s\tau}}(t) \tag{11} $$

$$ H(t, p_{\alpha s}, p_s, q_s, p_{s\tau}, q_{s\tau}) = p_s(t)\dot{q}_s(t) + p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^{\alpha}q_s(t) + p_{s\tau}(t)\dot{q}_{s\tau}(t) - L \tag{12} $$

则对（12）式进行分数阶广义动量进行求导，得到

$$ {}_{t_1}^{C}D_t^{\alpha}q_s(t) = \frac{\partial H}{\partial p_{\alpha s}}(t),\ \dot{q}_s(t) + \dot{q}_{s\tau}(t+\tau) = \frac{\partial H}{\partial p_s}(t) + \frac{\partial H}{\partial p_{s\tau}}(t+\tau),\quad t_1 \leqslant t \leqslant t_2 - \tau, $$

$$ {}_{t_1}^{C}D_t^{\alpha}q_s(t) = \frac{\partial H}{\partial p_{\alpha s}}(t), $$

$$ \dot{q}_s(t) = \frac{\partial H}{\partial p_s}(t),\quad t_2 - \tau < t \leqslant t_2 \tag{13} $$

非保守系统的分数阶 Hamilton 原理为

$$\int_{t_1}^{t_2}[\delta p_{\alpha s}\,{}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_{\alpha s}(t)\delta\,{}_{t_1}^{C}D_t^{\alpha}q_s+\delta p_s\dot{q}_s(t)+p_s(t)\delta\dot{q}_s+\delta p_{s\tau}\dot{q}_{s\tau}(t)+p_{s\tau}(t)\delta\dot{q}_{s\tau}-\frac{\partial H}{\partial q_s}(t)\delta q_s-\frac{\partial H}{\partial q_{s\tau}}(t)\delta q_{s\tau}-\frac{\partial H}{\partial p_{\alpha s}}(t)\delta p_{\alpha s}-\frac{\partial H}{\partial p_s}(t)\delta p_s-\frac{\partial H}{\partial p_{s\tau}}(t)\delta p_{s\tau}]\mathrm{d}t=0 \tag{14}$$

且满足条件

$$q_s(t)=\Omega_s(t),\quad t\in[t_1-\tau,t_1],\quad q_s(t)=q_s(t_2),\quad t=t_2,(s=1,2,\cdots,n) \tag{15}$$

其中 $\Omega_s(t)$ 为 $[t_1-\tau,t_1]$ 上的已知分段光滑函数，$q_s(t_2)$ 为已知实数。

类似于文献［20］的推导，由（13）和（14）式，可导出

$$-{}_tD_{t_2-\tau}^{\alpha}p_{\alpha s}(t)+\dot{p}_s(t)+\dot{p}_{s\tau}(t+\tau)+\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)+\frac{1}{\Gamma(\alpha)}{}_tD_{t_2-\tau}^{\alpha}\int_{t_2-\tau}^{t_2}({}_tD_{t_2}^{\alpha}\,p_{\alpha s}(z)(z-t)^{\alpha-1})\mathrm{d}z=0$$

$$t_1\leqslant t\leqslant t_2-\tau$$

$$-{}_tD_{t_2}^{\alpha}\,p_{\alpha s}(t)+\dot{p}_s(t)+\frac{\partial H}{\partial q_s}(t)=0$$

$$t_2-\tau<t\leqslant t_2 \tag{16}$$

联立方程（13）和（16）给出 Caputo 导数下的含时滞的力学系统的分数阶 Hamilton 正则方程。

3　含时滞的 Hamilton 系统的分数阶 Noether 对称性

Caputo 导数下的含时滞的 Hamilton 系统的分数阶 Hamilton 作用量表示为

$$S(\gamma)=\int_{t_1}^{t_2}(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)+p_{s\tau}(t)\dot{q}_{s\tau}(t)-H)\mathrm{d}t \tag{17}$$

其中 γ 为某已知曲线，引入无限小群变换

$$\bar{t}=t+\Delta t,\bar{q}_s(\bar{t})=q_s(t)+\Delta q_s,\bar{p}_s(\bar{t})=p_s(t)+\Delta p_s,(s=1,2,\cdots,n) \tag{18}$$

其展开式

$$\bar{t}=t+\varepsilon_\sigma\xi_0^\sigma(t,q_k,p_k,p_{\alpha k}),$$
$$\bar{q}_s(\bar{t})=q_s(t)+\varepsilon_\sigma\xi_s^\sigma(t,q_k,p_k,p_{\alpha k}),$$
$$\bar{p}_s(t)=p_s(t)+\varepsilon_\sigma\eta_s^\sigma(t,q_k,p_k,p_{\alpha k})$$
$$(s,k=1,2,\cdots,n) \tag{19}$$

其中 $\varepsilon_\sigma(\sigma=1,2,\cdots,r)$ 为无限小参数，ξ_0^σ，ξ_s^σ 和 η_s^σ 为无限小生成元或生成函数。在变换（18）下，含时滞的分数阶 Hamilton 作用量（17）变为

$$S(\bar{\gamma})=\int_{t_1}^{\bar{t}}[\bar{p}_{\alpha s}(\bar{t})\,{}_{t_1}^{C}D_t^{\alpha}\bar{q}_s(\bar{t})+\bar{p}_s(\bar{t})\dot{\bar{q}}_s(\bar{t})+\bar{p}_{s\tau}(\bar{t})\dot{\bar{q}}_{s\tau}(\bar{t})-H(\bar{t},\bar{p}_{\alpha s}(\bar{t}),\bar{p}_s(\bar{t}),\bar{q}_s(\bar{t}),\bar{p}_{s\tau}(\bar{t}),\bar{q}_{s\tau}(\bar{t}))]\mathrm{d}\bar{t} \tag{20}$$

其中 $\bar{\gamma}$ 为 γ 的邻近曲线。

假设 ΔS 是变换前后的差 $S(\bar{\gamma})-S(\gamma)$ 相对 ε 的主线性部分，则有

$$\Delta S=\int_{t_1}^{t_2}\Big[\Delta p_{\alpha s}\,{}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_{\alpha s}(t)\Delta\,{}_{t_1}^{C}D_t^{\alpha}q_s+\Delta p_s\dot{q}_s(t)+p_s(t)\Delta\dot{q}_s+\Delta p_{s\tau}\dot{q}_{s\tau}(t)+p_{s\tau}(t)\Delta\dot{q}_{s\tau}-\frac{\partial H}{\partial t}(t)\Delta t-\frac{\partial H}{\partial q_s}(t)\Delta q_s-\frac{\partial H}{\partial p_{\alpha s}}(t)\Delta p_{\alpha s}-\frac{\partial H}{\partial q_{s\tau}}(t)\Delta q_{s\tau}-\frac{\partial H}{\partial p_s}(t)\Delta p_s-\frac{\partial H}{\partial p_{s\tau}}(t)\Delta p_{s\tau}+(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)+p_{s\tau}(t)\dot{q}_{s\tau}(t)-H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\Big]\mathrm{d}t \tag{21}$$

由于

$$\int_{t_1}^{t_2}[\Delta p_{s\tau}\dot{q}_{s\tau}(t)+p_{s\tau}(t)\Delta\dot{q}_{s\tau}-\frac{\partial H}{\partial q_{s\tau}}(t)\Delta q_{s\tau}-\frac{\partial H}{\partial p_{s\tau}}(t)\Delta p_{s\tau}+p_{s\tau}(t)\dot{q}_{s\tau}(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)]\mathrm{d}t=\int_{t_1}^{t_2-\tau}[\Delta p_s(\theta)\dot{q}_{s\tau}(\theta+\tau)+p_{s\tau}(\theta+\tau)\Delta\dot{q}_s(\theta)-\frac{\partial H}{\partial q_{s\tau}}(\theta+\tau)\Delta q_s(\theta)-\frac{\partial H}{\partial p_{s\tau}}(\theta+\tau)\Delta p_s(\theta)+p_{s\tau}(\theta+\tau)\dot{q}_{s\tau}(\theta+\tau)\frac{\mathrm{d}}{\mathrm{d}\theta}(\Delta\theta)]\mathrm{d}\theta \tag{22}$$

将式（22）代入式（21），有

$$\Delta S=\int_{t_1}^{t_2-\tau}\Big[\Delta p_{\alpha s}\Big({}_{t_1}^{C}D_t^{\alpha}q_s(t)-\frac{\partial H}{\partial p_{\alpha s}}(t)\Big)+p_{\alpha s}(t)\Delta\,{}_{t_1}^{C}D_t^{\alpha}q_s+(p_s(t)+p_{s\tau}(t+\tau))\Delta\dot{q}_s+\Delta p_s\Big(\dot{q}_s(t)+\dot{q}_{s\tau}(t+\tau)-\frac{\partial H}{\partial p_s}(t)-\frac{\partial H}{\partial p_{s\tau}}(t+\tau)\Big)-\frac{\partial H}{\partial t}(t)\Delta t-\Big(\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\Big)\Delta q_s+\Big(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H\Big)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\Big]\mathrm{d}t+\int_{t_2-\tau}^{t_2}\Big[p_{\alpha s}(t)\Delta\,{}_{t_1}^{C}D_t^{\alpha}q_s+\Delta p_{\alpha s}(t)\Big({}_{t_1}^{C}D_t^{\alpha}q_s-\frac{\partial H}{\partial p_{\alpha s}}(t)\Big)+\Delta p_s\Big(\dot{q}_s(t)-\frac{\partial H}{\partial p_s}(t)\Big)+p_s(t)\Delta\dot{q}_s(t)-\frac{\partial H}{\partial t}(t)\Delta t-\frac{\partial H}{\partial q_s}(t)\Delta q_s+(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)-H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\Big]\mathrm{d}t \tag{23}$$

注意到

$$\delta q_s = \Delta q_s - \dot{q}_s \Delta t \delta p_s = \Delta p_s - \dot{p}_s \Delta t \Delta {}_{t_1}^{C}D_t^{\alpha} q_s = {}_{t_1}^{C}D_t^{\alpha} \delta q_s + \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}^{C}D_t^{\alpha} q_s\right)\Delta t \tag{24}$$

以及变换（18）和分部积分公式（6）、（7）、（8）和（9），则式（32）可化为

$$\begin{aligned}\Delta S = \int_{t_1}^{t_2-\tau} \varepsilon_\sigma \Big\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big[(p_s(t) + p_{s\tau}(t+\tau))\bar{\xi}_s^\sigma + \\ \xi_0^\sigma (p_{\alpha s}(t)\, {}_{t_1}^{C}D_t^{\alpha} q_s(t) + p_s(t)\dot{q}_s(t) + \\ p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau) - H) + \\ \int_{t_1}^{t}\Big({}_{t_1}^{C}D_\theta^{\alpha}\bar{\xi}_s^\sigma p_{\alpha s}(\theta) - \bar{\xi}_s^\sigma\, {}_\theta D_{t_2-\tau}^{\alpha} p_{\alpha s}(\theta) + \\ \frac{1}{\Gamma(\alpha)}\bar{\xi}_s^\sigma\, {}_\theta D_{t_2-\tau}^{\alpha}\int_{t_2-\tau}^{t_2}({}_\theta D_{t_2}^{\alpha} p_{\alpha s}(z)(z-\theta)^{\alpha-1}\mathrm{d}z\Big)\mathrm{d}\theta\Big] + \\ \bar{\xi}_s^\sigma\Big[{}_{t_1}^{C}D_{t_2-\tau}^{\alpha} p_{\alpha s}(t) - \dot{p}_s(t) - \dot{p}_{s\tau}(t+\tau) - \\ \frac{\partial H}{\partial q_s}(t) - \frac{\partial H}{\partial q_{s\tau}}(t+\tau) - \frac{1}{\Gamma(\alpha)}\, {}_tD_{t_2-\tau}^{\alpha} \\ \int_{t_2-\tau}^{t_2}({}_tD_{t_2}^{\alpha} p_{\alpha s}(z)(z-t)^{\alpha-1})\mathrm{d}z\Big] + \\ \bar{\eta}_{\alpha s}^\sigma\Big({}_{t_1}D_t^{\alpha} q_s(t) - \frac{\partial H}{\partial p_{\alpha s}}(t)\Big) + \bar{\eta}_s^\sigma\Big(\dot{q}_{s\tau}(t+\tau) + \\ \dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\Big)\Big\}\mathrm{d}t + \\ \int_{t_2-\tau}^{t_2}\varepsilon_\sigma\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big[p_s(t)\bar{\xi}_s^\sigma + \int_{t_1}^{t}({}_{t_1}^{C}D_\theta^{\alpha}\bar{\xi}_s^\sigma p_{\alpha s}(\theta) - \\ \bar{\xi}_s^\sigma\, {}_\theta D_{t_2}^{\alpha} p_{\alpha s}(\theta))\mathrm{d}\theta + (p_{\alpha s}(t)\, {}_{t_1}^{C}D_t^{\alpha} q_s(t) + \\ p_s(t)\dot{q}_s(t) - H)\xi_0^\sigma \Big] + \bar{\xi}_s^\sigma\Big[{}_tD_{t_2}^{\alpha} p_{\alpha s}(t) - \\ \dot{p}_s(t) - \frac{\partial H}{\partial q_s}(t)\Big] + \bar{\eta}_{\alpha s}^\sigma\Big({}_{t_1}^{C}D_t^{\alpha} q_s(t) - \frac{\partial H}{\partial p_{\alpha s}}(t)\Big) + \\ \bar{\eta}_s^\sigma\Big(\dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t)\Big)\Big\}\mathrm{d}t\end{aligned} \tag{25}$$

其中

$$\bar{\xi}_s^\sigma = \xi_s^\sigma - \dot{q}_s\xi_0^\sigma,\ \bar{\eta}_s^\sigma = \eta_s^\sigma - \dot{p}_s\xi_0^\sigma,\ \bar{\eta}_{\alpha s}^\sigma = \eta_{\alpha s}^\sigma - \dot{p}_{\alpha s}\xi_0^\sigma \tag{26}$$

式（23）和式（25）称为 Caputo 导数下的含时滞的力学系统的分数阶 Hamilton 作用量变分的基本公式。

如果成立

$$\Delta S = 0 \tag{27}$$

则称无限小变换（18）是含时滞的分数阶 Noether 意义下的对称变换。此时，由式（23）得

当 $t_1 \leqslant t \leqslant t_2 - \tau$ 时，有

$$\begin{aligned}\Delta p_{\alpha s}\Big({}_{t_1}^{C}D_t^{\alpha} q_s(t) - \frac{\partial H}{\partial p_{\alpha s}}(t)\Big) + \\ \Delta p_s\Big(\dot{q}_s(t) + \dot{q}_{s\tau}(t+\tau) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\Big) + \\ p_{\alpha s}(t)\Delta\, {}_{t_1}^{C}D_t^{\alpha} q_s + p_s(t) + p_{s\tau}(t+\tau))\Delta\dot{q}_s - \\ \frac{\partial H}{\partial t}(t)\Delta t - \Big(\frac{\partial H}{\partial q_s}(t) + \frac{\partial H}{\partial q_{s\tau}}(t+\tau)\Big)\Delta q_s + \\ (p_{\alpha s}(t)\, {}_{t_1}^{C}D_t^{\alpha} q_s(t) + p_s(t)\dot{q}_s(t) + \\ p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau) - H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0\end{aligned} \tag{28}$$

当 $t_2 - \tau < t \leqslant t_2$ 时，有

$$\begin{aligned}\Delta p_{\alpha s}\Big({}_{t_1}^{C}D_t^{\alpha} q_s(t) - \frac{\partial H}{\partial p_{\alpha s}}(t)\Big) + p_{\alpha s}(t)\Delta\, {}_{t_1}^{C}D_t^{\alpha} q_s + \\ \Delta p_s\Big(\dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t)\Big) + p_s(t)\Delta\dot{q}_s - \\ \frac{\partial H}{\partial t}(t)\Delta t - \frac{\partial H}{\partial q_s}(t)\Delta q_s + (p_{\alpha s}(t)\, {}_{t_1}D_t^{\alpha} q_s(t) + \\ p_s(t)\dot{q}_s(t) - H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0\end{aligned} \tag{29}$$

式（28）和（29）可化为，当 $t_1 \leqslant t \leqslant t_2 - \tau$ 时，有

$$\begin{aligned}p_{\alpha s}(t)\Big({}_{t_1}^{C}D_t^{\alpha}\bar{\xi}_s^\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_{t_1}D_t^{\alpha} q_s\xi_0^\sigma\Big) + \\ (p_s(t) + p_{s\tau}(t+\tau))\dot{\xi}_s^\sigma - \frac{\partial H}{\partial t}(t)\xi_0^\sigma - \\ \Big(\frac{\partial H}{\partial q_s}(t) + \frac{\partial H}{\partial q_{s\tau}}(t+\tau)\Big)\xi_s^\sigma + \\ (p_{\alpha s}(t)\, {}_{t_1}^{C}D_t^{\alpha} q(t) - H)\dot{\xi}_0^\sigma = 0 \\ (\sigma = 1,2,\cdots,r)\end{aligned} \tag{30}$$

当 $t_2 - \tau < t \leqslant t_2$ 时，有

$$\begin{aligned}p_{\alpha s}(t)\Big({}_{t_1}^{C}D_t^{\alpha}\bar{\xi}_s^\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\, {}_{t_1}D_t^{\alpha} q_s\xi_0^\sigma\Big) + \\ p_s(t)\dot{\xi}_s^\sigma - \frac{\partial H}{\partial t}(t)\xi_0^\sigma - \frac{\partial H}{\partial q_s}(t)\xi_s^\sigma + \\ (p_{\alpha s}(t)\, {}_{t_1}^{C}D_t^{\alpha} q_s(t) - H)\dot{\xi}_0^\sigma = 0 \\ (\sigma = 1,2,\cdots,r)\end{aligned} \tag{31}$$

当 $r = 1$ 时，式（30）和（31）称为含时滞的 Hamilton 系统的分数阶 Noether 等式。

如果成立

$$\Delta S = -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\mathrm{d}t \tag{32}$$

其中 $G = G(t, q_s, p_{\alpha s}, p_s, q_{s\tau}, p_{s\tau})$ 为规范函数，则称无限小变换（18）是含时滞的分数阶 Noether 意义下的准对称变换。此时，由式（23）得

当 $t_1 \leqslant t \leqslant t_2 - \tau$ 时，有

$$\Delta p_{\alpha s}\Big({}_{t_1}^{C}D_t^{\alpha} q_s(t) - \frac{\partial H}{\partial p_{\alpha s}}(t)\Big) + \Delta p_s\Big(\dot{q}_s(t) + \dot{q}_{s\tau}(t+\tau) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\Big) +$$

$$p_{\alpha s}(t)\Delta_{t_1}^{C}D_t^{\alpha}q_s+(p_s(t)+p_{s\tau}(t+\tau))\Delta\dot{q}_s-\frac{\partial H}{\partial t}(t)\Delta t-\left(\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right)\Delta q_s+(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}\Delta G \tag{33}$$

当 $t_2-\tau<t\leqslant t_2$ 时，有

$$\Delta p_{\alpha s}\left({}_{t_1}^{C}D_t^{\alpha}q_s(t)-\frac{\partial H}{\partial p_{\alpha s}}(t)\right)+p_{\alpha s}(t)\Delta_{t_1}^{C}D_t^{\alpha}q_s+\Delta p_s\left(\dot{q}_s(t)-\frac{\partial H}{\partial p_s}(t)\right)+p_s(t)\Delta\dot{q}_s-\frac{\partial H}{\partial t}(t)\Delta t-\frac{\partial H}{\partial q_s}(t)\Delta q_s+(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)-H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}\Delta G \tag{34}$$

式（33）和（34）可化表示为，当 $t_1\leqslant t\leqslant t_2-\tau$ 时，有

$$p_{\alpha s}(t)\left({}_{t_1}^{C}D_t^{\alpha}\bar{\xi}_s^{\sigma}+\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_s\xi_0^{\sigma}\right)+(p_s(t)+p_{s\tau}(t+\tau))\dot{\xi}_s^{\sigma}-\frac{\partial H}{\partial t}(t)\xi_0^{\sigma}-\left(\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right)\xi_s^{\sigma}+(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q(t)-H)\dot{\xi}_0^{\sigma}=-G^{\sigma}\quad(\sigma=1,2,\cdots,r) \tag{35}$$

当 $t_2-\tau<t\leqslant t_2$ 时，有

$$p_{\alpha s}(t)\left({}_{t_1}^{C}D_t^{\alpha}\bar{\xi}_s^{\sigma}+\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_s\xi_0^{\sigma}\right)+p_s(t)\dot{\xi}_s^{\sigma}-\frac{\partial H}{\partial t}(t)\xi_0^{\sigma}-\frac{\partial H}{\partial q_s}(t)\xi_s^{\sigma}+(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)-H)\dot{\xi}_0^{\sigma}=-G^{\sigma}\quad(\sigma=1,2,\cdots,r) \tag{36}$$

其中 $\Delta G=\varepsilon_{\sigma}G^{\sigma}$，当 $r=1$ 时，式（35）和（36）也称为含时滞的 Hamilton 系统的分数阶 Noether 等式。

4　含时滞的 Hamilton 系统的分数阶 Noether 理论

函数

$$I(t,t+\tau,q_s,q_{s\tau},q_s(t+\tau),p_{\alpha s},p_{s\tau},p_s,p_s(t+\tau),p_{\alpha s}(t+\tau))$$

称为含时滞的力学系统的分数阶守恒量，当且仅当沿着运动方程（13）和（16）的解曲线恒成立

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t,t+\tau,q_s,q_{s\tau},q_s(t+\tau),p_{\alpha s},p_{s\tau},p_s,p_s(t+\tau),p_{\alpha s}(t+\tau))=0 \tag{37}$$

对于含时滞的分数阶 Hamilton 系统（13）和（16），如果能找到系统的分数阶 Noether 对称变换或 Noether 准对称变换，便可求得相应的分数阶守恒量。于是，有如下定理

定理 1　对于含时滞的分数阶 Hamilton 系统（13）和（16），如果无限小变换（18）是系统的分数阶 Noether 对称变换，则系统存在 r 个线性独立的分数阶守恒量，当 $t_1\leqslant t\leqslant t_2-\tau$ 时，形如

$$I^{\sigma}=(p_s(t)+p_{s\tau}(t+\tau))\xi_s^{\sigma}+\int_{t_1}^{t}\left[{}_{t_1}^{C}D_{\theta}^{\alpha}\bar{\xi}_s^{\sigma}p_{\alpha s}(\theta)-\bar{\xi}_s^{\sigma}{}_{\theta}D_{t_2-\tau}^{\alpha}p_{\alpha s}(\theta)+\frac{1}{\Gamma(\alpha)}\bar{\xi}_{s}^{\sigma}{}_{\theta}D_{t_2-\tau}^{\alpha}\int_{t_2-\tau}^{t_2}({}_{\theta}D_{t_2}^{\alpha}p_{\alpha s}(z)(z-\theta)^{\alpha-1})\mathrm{d}z\right]\mathrm{d}\theta+(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)-H)\xi_0^{\sigma}=\mathrm{const.}\quad(\sigma=1,2,\cdots,r) \tag{38}$$

当 $t_2-\tau<t\leqslant t_2$ 时，形如

$$I^{\sigma}=p_s(t)\xi_s^{\sigma}+\int_{t_1}^{t}({}_{t_1}^{C}D_{\theta}^{\alpha}\bar{\xi}_s^{\sigma}p_{\alpha s}(\theta)-\bar{\xi}_s^{\sigma}{}_{\theta}D_{t_2}^{\alpha}p_{\alpha s}(\theta))\mathrm{d}\theta+\xi_0^{\sigma}(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)-H)=\mathrm{const.}\quad(\sigma=1,2,\cdots,r) \tag{39}$$

证明：将系统的分数阶 Hamilton 正则方程代入式（25），并考虑到 ε_{σ} 的独立性和积分区间的任意性，当 $t_1\leqslant t\leqslant t_2-\tau$ 时，得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(p_s(t)+p_{s\tau}(t+\tau))\xi_s^{\sigma}+\int_{t_1}^{t}\Big[{}_{t_1}^{C}D_{\theta}^{\alpha}\bar{\xi}_s^{\sigma}p_{\alpha s}(\theta)-\bar{\xi}_s^{\sigma}{}_{\theta}D_{t_2-\tau}^{\alpha}p_{\alpha s}(\theta)+\frac{1}{\Gamma(\alpha)}\bar{\xi}_s^{\sigma}{}_{\theta}^{\alpha}D_{t_2-\tau}\int_{t_2-\tau}^{t_2}({}_{\theta}D_{t_2}^{\alpha}p_{\alpha s}(z)(z-\theta)^{\alpha-1})\mathrm{d}z\Big]\mathrm{d}\theta+\xi_0^{\sigma}(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H)\Big\}=$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(p_s(t)+p_{s\tau}(t+\tau))\bar{\xi}_s^{\sigma}+\int_{t_1}^{t}\Big[{}_{t_1}^{C}D_{\theta}^{\alpha}\bar{\xi}_s^{\sigma}p_{\alpha s}(\theta)-\bar{\xi}_s^{\sigma}{}_{\theta}^{\alpha}D_{t_2-\tau}p_{\alpha s}(\theta)+\frac{1}{\Gamma(\alpha)}\bar{\xi}_s^{\sigma}{}_{\theta}^{\alpha}D_{t_2-\tau}\int_{t_2-\tau}^{t_2}({}_{\theta}D_{t_2}^{\alpha}p_{\alpha s}(z)(z-\theta)^{\alpha-1})\mathrm{d}z\Big]\mathrm{d}\theta+\xi_0^{\sigma}(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)-H)\Big\}=0,\quad(\sigma=1,2,\cdots,r) \tag{40}$$

当 $t_2-\tau<t\leqslant t_2$ 时，有

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[p_s(t)\bar{\xi}_s^{\sigma}+\int_{t_1}^{t}({}_{t_1}^{C}D_{\theta}^{\alpha}\bar{\xi}_s^{\sigma}p_{\alpha s}(\theta)-\bar{\xi}_s^{\sigma}{}_{\theta}D_{t_2}^{\alpha}p_{\alpha s}(\theta))\mathrm{d}\theta+\xi_0^{\sigma}(p_{\alpha s}(t){}_{t_1}^{C}D_t^{\alpha}q_s(t)+p_s(t)\dot{q}_s(t)-H)\Big]=$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\ p_s(t)\xi_s^\sigma + \int_{t_1}^{t}\big({}_{t_1}^{C}D_\theta^\alpha\bar{\xi}_s^\sigma p_{\alpha s}(\theta) - \bar{\xi}_s^\sigma{}_\theta D_{t_2}^\alpha p_{\alpha s}(\theta)\big)\mathrm{d}\theta + \xi_0^\sigma\big(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^\alpha q_s(t) - H\big)\Big] = 0,$$

$$(\sigma = 1,2,\cdots,r) \tag{41}$$

将式（40）和（41）积分，便得到结果。

定理 2 对于含时滞的分数阶 Hamilton 系统（13）和（16），如果无限小变换（18）是系统的分数阶 Noether 准对称变换，则系统存在 r 个线性独立的分数阶守恒量，当 $t_1 \leqslant t \leqslant t_2 - \tau$ 时，形如

$$I^\sigma = (p_s(t) + p_{s\tau}(t+\tau))\xi_s^\sigma + \int_{t_1}^{t}\Big[\ {}_{t_1}^{C}D_\theta^\alpha\bar{\xi}_s^\sigma p_{\alpha s}(\theta) - \bar{\xi}_s^\sigma{}_\theta^\alpha D_{t_2-\tau}p_{\alpha s}(\theta) + \frac{1}{\Gamma(\alpha)}\bar{\xi}_s^\sigma\ {}_\theta^\alpha D_{t_2-\tau}\int_{t_2-\tau}^{t_2}\big({}_\theta D_{t_2}^\alpha p_{\alpha s}(z)(z-\theta)^{\alpha-1}\big)\mathrm{d}z\Big]\mathrm{d}\theta + \Big(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^\alpha q_s(t) - H\Big)\xi_0^\sigma + G^\sigma = \text{const.}$$

$$(\sigma = 1,2,\cdots,r) \tag{42}$$

当 $t_2 - \tau < t \leqslant t_2$ 时，形如

$$I^\sigma = p_s(t)\xi_s^\sigma + \int_{t_1}^{t}\Big({}_{t_1}^{C}D_\theta^\alpha\bar{\xi}_s^\sigma p_{\alpha s}(\theta) - \bar{\xi}_s^\sigma{}_\theta D_{t_2}^\alpha p_{\alpha s}(\theta)\Big)\mathrm{d}\theta + \xi_0^\sigma\Big(p_{\alpha s}(t)\,{}_{t_1}^{C}D_t^\alpha q_s(t) - H\Big) + G^\sigma = \text{const.}$$

$$(\sigma = 1,2,\cdots,r) \tag{43}$$

证明：结合式（25）和（32），并利用分数阶正则方程（13）和（16），且考虑到 ε_σ 的独立性和积分区间的任意性，即得结论。

定理 1 和定理 2 称为含时滞的 Hamilton 系统的分数阶 Noether 定理。由分数阶 Noether 定理可知，对于 Caputo 导数下的含时滞的 Hamilton 系统，如果能找到一个含时滞的分数阶 Noether 对称变换或分数阶 Noether 准对称变换，便可能得到力学系统的一个分数阶守恒量。

5 算 例

例 已知力学系统的 Lagrange 函数为

$$L = \frac{1}{2}\Big[\Big({}_{t_1}^{C}D_t^\alpha q(t)\Big)^2 + \dot{q}^2(t)\Big] - \frac{1}{2}q^2(t-\tau) \tag{44}$$

其中 k 为常数，时滞常量 $\tau < t_2 - t_1$ 为已知正实数。并满足条件：当时，$q(t) = \Omega(t)$，这里 $\Omega(t)$ 是区间 $[t_1 - \tau, t_1]$ 上的已知分段光滑函数；当 $t = t_2$ 时，$t \in [t_1 - \tau, t_1]q(t) = q(t_2)$，这里 $q(t_2)$ 是某一确定值。

由式（11）和（12），得到

$$p(t) = \dot{q}(t)p_\alpha(t) = {}_{t_1}^{C}D_t^\alpha q(t)$$

$$H = \frac{1}{2}[p_\alpha^2(t) + p^2(t)] + \frac{1}{2}q^2(t-\tau) \tag{45}$$

则系统的正则方程为

$$\begin{gathered}{}_{t_1}^{C}D_t^\alpha q(t) - p_\alpha(t) = 0\dot{q}(t) - p(t) = 0 - {}_tD_{t_2-\tau}^\alpha p_\alpha(t) + p(t) + q_\tau(t+\tau) + \frac{1}{\Gamma(\alpha)}\,{}_tD_{t_2-\tau}^\alpha\int_{t_2-\tau}^{t_2}\big({}_tD_{t_2}^\alpha p_{\alpha s}(z)(z-t)^{\alpha-1}\big)\mathrm{d}z = 0\\ t_1 \leqslant t \leqslant t_2 - \tau\\ {}_{t_1}^{C}D_t^\alpha q(t) - p_\alpha(t) = 0\\ \dot{q}(t) - p(t) = 0\\ -{}_tD_{t_2}^\alpha p_\alpha(t) + p(t) = 0\\ t_2 - \tau < t \leqslant t_2\end{gathered} \tag{46}$$

分数阶 Noether 等式（35）和（36），得到

$$\begin{gathered}p_\alpha(t)\Big({}_{t_1}^{C}D_t^\alpha\bar{\xi}_1 + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{C}D_t^\alpha q\xi_0\Big) + p(t)\dot{\xi}_1 - q_\tau(t+\tau)\xi_1 + (p_\alpha(t)\,{}_{t_1}^{C}D_t^\alpha q(t) - H)\dot{\xi}_0 = -\dot{G}\\ t_1 \leqslant t \leqslant t_2 - \tau\\ p_\alpha(t)\Big({}_{t_1}^{C}D_t^\alpha\bar{\xi}_1 + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{C}D_t^\alpha q\xi_0\Big) + p(t)\dot{\xi}_1 + (p_\alpha(t)\,{}_{t_1}^{C}D_t^\alpha q(t) - H)\dot{\xi}_0 = -\dot{G}\\ t_2 - \tau < t \leqslant t_2\end{gathered} \tag{47}$$

方程（47）有解

$$\xi_0^1 = 1, \xi_1^1 = 0, G^1 = 0, t_1 \leqslant t \leqslant t_2 \tag{48}$$

$$\begin{gathered}\xi_0^2 = 1, \xi_1^2 = \dot{q}(t),\\ G^2 = -\frac{1}{2}[(p_\alpha(t))^2 + p^2(t) - q_\tau^2(t+\tau)],\\ t_1 \leqslant t < t_2 - \tau,\ \xi_0 = 1, \xi_1 = \dot{q}(t),\\ G = -\frac{1}{2}[(p_\alpha(t))^2 + p^2(t)],\\ t_2 - \tau \leqslant t \leqslant t_2\end{gathered} \tag{49}$$

生成元（48）和（49）得到相应于系统的分数阶 Noether 对称性。由定理 1 和定理 2，得到

$$\begin{gathered}I^1 = \int_{t_1}^{t}\Big[-{}_{t_1}^{C}D_\theta^\alpha\dot{q}(\theta)p_\alpha(\theta) + \dot{q}(\theta)\,{}_\theta^\alpha D_{t_2-\tau}p_\alpha(\theta) - \frac{1}{\Gamma(\alpha)}\dot{q}\,{}_\theta^\alpha D_{t_2-\tau}\int_{t_2-\tau}^{t_2}\big({}_\theta D_{t_2}^\alpha p_{\alpha s}(z)(z-\theta)^{\alpha-1}\big)\mathrm{d}z\Big]\mathrm{d}\theta + p_\alpha(t)\,{}_{t_1}^{C}D_t^\alpha q(t) - \frac{1}{2}\Big[p_\alpha^2(t) + p^2(t) + q^2(t-\tau)\Big] = \text{const.}\quad t_1 \leqslant t \leqslant t_2 - \tau,\\ I^1 = \int_{t_1}^{t}\Big(-{}_{t_1}^{C}D_\theta^\alpha\dot{q}(\theta)p_\alpha(\theta) + \dot{q}(\theta)\,{}_\theta D_{t_2}^\alpha p_\alpha(\theta)\Big)\mathrm{d}\theta + p_\alpha(t)\,{}_{t_1}^{C}D_t^\alpha q(t) - \frac{1}{2}\Big[p_\alpha^2(t) + p^2(t) + q^2(t-\tau)\Big] = \text{const.}\quad t_2 - \tau < t \leqslant t_2\end{gathered} \tag{50}$$

$$I^2 = (p(t) + p_\tau(t+\tau))\dot{q}(t) + p_\alpha(t)\,{}_{t_1}^{C}D_t^\alpha q(t) - p^2(t) - (p_\alpha(t))^2 - \frac{1}{2}\left[q^2(t-\tau) - q_\tau^2(t+\tau)\right],\quad t_1 \leqslant t < t_2 - \tau$$

$$I^2 = p(t)\dot{q}(t) + p_\alpha(t)\,{}_{t_1}^{C}D_t^\alpha q(t) - p^2(t) - (p_\alpha(t))^2 - \frac{1}{2}q^2(t-\tau),\quad t_2 - \tau \leqslant t \leqslant t_2 \tag{51}$$

式（50）和（51）为含时滞的 Hamilton 系统相应于分数阶 Noether 对称性（48）和 Noether 准对称性（49）的分数阶守恒量。

若分数阶导数项不存在时，守恒量（50）和（51）就成为

$$I^1 = -\frac{1}{2}\left[p^2(t) + p^2(t) + kq^2(t-\tau)\right] = \text{const.}\quad t_1 \leqslant t \leqslant t_2 \tag{52}$$

$$I^2 = (p(t) + p_\tau(t+\tau))\dot{q}(t) - p^2(t) - \frac{1}{2}\left[q^2(t-\tau) - q_\tau^2(t+\tau)\right],\quad t_1 \leqslant t < t_2 - \tau$$

$$I^2 = p(t)\dot{q}(t) - p^2(t) - \frac{1}{2}q^2(t-\tau),\quad t_2 - \tau \leqslant t \leqslant t_2 \tag{53}$$

式（52）和（53）是含时滞的 Hamilton 系统的 Noether 对称性和准对称性的守恒量。如果时滞常量 τ 不存在，则式（52）和（53）就简化为经典力学系统的守恒量。

6　结　论

本文基于 Caputo 导数建立了含时滞的 Hamilton 系统的分数阶 Hamilton 原理，导出了含时滞的 Hamilton 系统的分数阶 Hamilton 正则方程（13）和（16），得到了含时滞的分数阶 Hamilton 作用量变分的两个基本公式（23）和（25），得到了含时滞的分数阶 Noether 对称性；建立了含时滞的 Hamilton 系统的分数阶 Noether 理论。文章的方法和结果具有普遍适用性，可应用于分数阶模型下的含时滞的最优控制系统，也可应用于含时滞的各种约束力学系统。

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